Internal Forces in Fluids

A kinder and gentler approach

We will consider two types of internal forces, those that arise because the fluid particles experience nonuniform distribution of pressure, and those that arise because of molecular friction between individual particles. We will refer to the former by **pressure gradients** and the latter by **viscous forces**.

As its name suggests, a pressure gradient force is mathematically represented by ∇p, where p is a smooth function of space and time. In cartesian coordinates ∇p is written as

$$\nabla p = \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \rangle. \tag{1}$$

The dimensions of p are force per unit area with units such as psi (pounds per square inches). Because the operation of gradient introduces a dimension of length to the denominator of the dimension of p (why?), the dimensions of ∇p are force per unit volume.

Recall that ∇p represents a vector whose potential is p and that $\nabla \times \nabla p = \mathbf{0}$. Also, recall that ∇p is perpendicular to curves (or surfaces) of constant pressure, namely, the isobars. A pressure gradient force is the type of a force that a fluid particle experiences when it senses compression or expansion in its immediate neighborhood. In addition to pressure gradients, typically a fluid particle experiences a shearing force, or a viscous force, due to interparticle friction.

The mathematical representation of viscous force is a little trickier. To build a bit of intuition as to what parameters viscous forces depend on, we consider the situation is Figure 1 where a force $\mathbf{F}$ is applied to the plate D located on top of a viscous fluid of depth L. One may ask what F, where F is the magnitude of $\mathbf{F}$, produces a linear horizontal velocity profile u shown in Figure 1 where $u(L) = u_0$ and $u(0) = 0$. Experimental evidence shows that F is directly proportional to u_0, to A, the area of the plate D, and inversely proportional to L, i.e.,

$$F = \mu \frac{A u_0}{L}.$$

The constant of proprtionality μ is called the (dynamic) coefficient of viscosity of the fluid. So the force per unit area due to the shearing motion described in Figure 1 is

$$\frac{F}{A} = \mu \frac{u_0}{l}. \tag{2}$$

Problem: What are the dimensions of μ?

The force $\mathbf{F}$ that led to the shearing motion in Figure 1 induces a particle P located at (x_0, y_0, z_0) to move parallel to the x-axis. Generally, the motions that we will study are endowed with more structure than the uniform shearing motion shown in Figure 1. In a somewhat more complicated motion, forces acting on P at an instant of time t_0 have the tendency to move P in an arbitrary but fixed direction for $t > t_0$. This direction may perhaps be different from the direction of the x-axis. A particle P' near P in the same motion is experiencing a similar force (but not necessarily an identical force), at time t_0 inducing it to move in a particular direction, perhaps a direction that is not parallel to that of P. This nonuniformity in the forces P and P' experience at time t_0 is not without its impact on the subsequent motion of these particles. In fact it leads to compressing and stretching of various filaments of fluid particles, resulting in a distribution of pressure gradient and viscous forces in the body of fluid. The motion of a body of fluid is best imagined as a consequence of the imbalance of forces in this distribution.

What if the velocity profile in the fluid is nonlinear, as would be the case when wind shears the surface of a choppy body of water? How are F and u related then? An important **assumption** of fluid and ocean dynamics suggests that the force per unit area that generates a nonlinear (or nonuniform) velocity field u is related to u by

$$\frac{F}{A} = \mu \frac{\partial u}{\partial z}. \tag{3}$$

Note the similarity between (2) and (3) and, in particular, the fact that (2) follows from (3) when $u(z) = u_0 \frac{z}{L}$ with u_0 a constant. The mathematical assumption that leads to (3) has withstood the test of many experiments and several years of mathematical modeling and has therefore been adopted by the great majority of practitioners of fluid dynamics. It must, however, be kept in mind that there are several schools of thought around the world that do not regard (3) as a realistic feature of certain fluid flows, especially those in which the flows are turbulent.

The force per unit area described in (3) is often denoted by

$$\tau_{zx} \tag{4}$$

and is called the zx component of the **stress** induced in the fluid due to the motion. It should be clear from Figure 1 why the indices z and x are used in (4).

How is one to produce a force per unit **volume** out of (3) to compare with the pressure gradient force (which has dimensions of force per unit volume) that a particle experiences? One could artificially introduce a length dimension to the denominator of (3), but that would not be a satisfactory way of developing a robust theory. Instead let us consider the situation in Figure 2.

In Figure 2 region D is a subbody of a fluid in the shape of a cube, centered at the point (x_0, y_0, z_0), with sides parallel to the coordinates planes and measuring Δx, Δy and Δz. We assume, much like the situation in the thought experiment that led to (2), that the faces represented by $z = z_0 + \frac{\Delta z}{2}$ and $z = z_0 - \frac{\Delta z}{2}$ are moving in the positive x direction. Each face is moving because a force per unit **area**, as described in (3), is acting on that face. According to (3), these are

$$\mu \frac{\partial u}{\partial z}\Big|_{z=z_0+\frac{\Delta z}{2}}, \quad \text{and } \mu \frac{\partial u}{\partial z}\Big|_{z=z_0-\frac{\Delta z}{2}}. \tag{5}$$

What is the impact of (5) on the point (x_0, y_0, z_0)? What is the resultant viscous force per unit **volume** at this point?

The resultant force on a particular face of our cube depends on the normal vector to that face; For instance, the resultant force on the face $z = z_0 + \frac{\Delta z}{2}$ is

$$\tau_{zx}(x_0, y_0, z_0 + \frac{\Delta z}{2})$$

while the resultant force on the face $z = z_0 - \frac{\Delta z}{2}$ is

$$-\tau_{zx}(x_0, y_0, z_0 - \frac{\Delta z}{2}).$$

The minus sign in $-\tau_{zx}$ reflects the fact that the normal to the subbody D on the face $z = z_0 - \frac{\Delta z}{2}$ points in the negative z direction.

So the reultant viscous force at (x_0, y_0, z_0) due to the shearing at the two faces $z = z_0 \pm \frac{\Delta z}{2}$ is

$$\left(\tau_{zx}(x_0, y_0, z_0 + \frac{\Delta z}{2}) - \tau_{zx}(x_0, y_0, z_0 - \frac{\Delta z}{2})\right) \Delta x \Delta y. \tag{6}$$

(Why is the term $\Delta x \Delta y$ present in this expression?) Expression (6) can be simplified considerably using Taylor's formula (recall that $f(z_0 + h) = f(z_0) + hf'(z_0) +$ higher order terms in h.) We have

$$\tau_{zx}(x_0, y_0, z_0 + \frac{\Delta z}{2}) = \tau_{zx}(x_0, y_0, z_0) + \frac{\Delta z}{2}\frac{\partial \tau_{zx}}{\partial z}\Big|_{(x_0,y_0,z_0)}$$

plus high order terms in Δz. Using this fact, and its equivalent expression for $\tau(x_0, y_0, z_0 - \frac{\Delta z}{2})$, we have

$$\text{resultant viscous force at } (x_0, y_0, z_0) = (\tau_{zx}(x_0, y_0, z_0) +$$

$$\frac{\Delta z}{2}\frac{\partial \tau_{zx}}{\partial z}|_{(x_0,y_0,z_0)})\Delta x \Delta y - (\tau_{zx}(x_0, y_0, z_0) - \frac{\Delta z}{2}\frac{\partial \tau_{zx}}{\partial z}|_{(x_0,y_0,z_0)})\Delta x \Delta y \tag{7}$$

which simplifies to

$$\frac{\partial \tau_{zx}}{\partial z}\Delta x \Delta y \Delta z. \tag{8}$$

This derivation suggests that

$$\frac{\partial \tau_{zx}}{\partial z} \tag{9}$$

is the shearing force (per unit volume) experienced at (x_0, y_0, z_0). Substituting (3) in (9) we have

$$\frac{\partial}{\partial z}(\mu \frac{\partial u}{\partial z}) \tag{10}$$

as the contribution of viscous shearing. When μ is constant, this expression reduces to

$$\mu \frac{\partial^2 u}{\partial z^2}.$$

In a somewhat more complicated flow where the point (x_0, y_0, z_0) experiences forces from all three directions, the resultant force per unit volume in the x-direction is

$$-\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + F_1 \tag{11}$$

where F_1 is the horizontal component of the resultant force corresponding to the external forces that act on a body, such as the Coriolis force, the gravitational force (the fluid weight), and the planetary gravitational force. The resultant force in (11) is balanced by density times acceleration

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}).$$

We can now write the Navier-Stokes equations of motion:

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + F_1,$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) + F_2, \qquad (12)$$
$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + F_3.$$

The equations in (12) represent one of the most important sets of equations in mathematical physics. These equations are expected to model motions as diverse as the Gulf and Jet streams, hurricanes and tornadoes, Hadley cells and modons, as well as the motion of a drop of rain on the windshield of your car, and how blood flows in our veins. It is one of the greatest triumphs of mathematics and physics that after nearly two hundreds have gone by since these equations were first written down, they still remain the most credible mathematical model we have for fluid flows.

Next we will explore what form the external force **F** should take to take into account the Coriolis force.

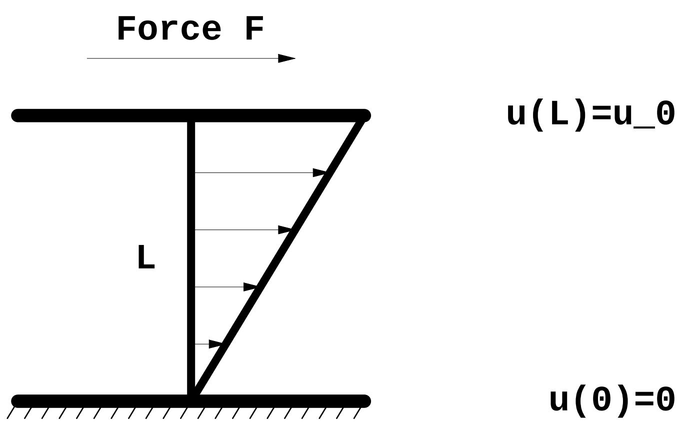

Figure 1: The upper plate, Plate D, is put into motion by applying force F. The fluid has depth L and crosssectional area A (not shown in the figure). The lower plate is kept stationary (hence, $u(0) = 0$). Once the flow reaches its steady-state motion, the upper plate will be moving with speed u_0 (hence, $u(L) = u_0$). Experimental evidence shows that F, u_0, A and μ, the coefficient of viscosity of the fluid, must be related according to $F = \mu\frac{Au_0}{L}$.